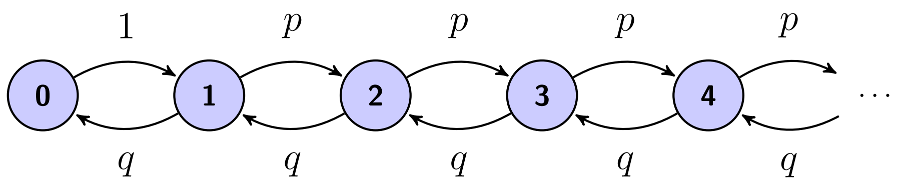
Exercise 1. Let $(X_n)_{n\geq 0}$ be an irreducible Markov chain with transition matrix P. For a fixed state k, let us denote $\bar{P}^{(k)}$ the matrix obtained from P by suppressing the line k and colon k. Then the states (X_n) are recurrent if and only if

$$\begin{cases} \bar{P}^{(k)}\boldsymbol{x} = \boldsymbol{x} \\ 0 \leq x_i \leq 1 \ \forall i \end{cases}$$

has only the null vector as a solution.
Use this result to discuss the nature of the states of the reflected random walk represented by the graph ($p + q = 1$):



Exercise 2. $M/M/1/\infty$ queue
Let us suppose that the arrivals at the EPFL service desk follow a Poisson process with parameter λ. When a client arrives, its service starts immediately if the desk is free. Otherwise, he waits for his turn. A queue with infinite length is allowed.
We assume that the service time of one customer follow an exponential distribution with parameter μ, the service duration of one customer is independent from the one of the others and independent from the Poisson process of arrivals.
Let us consider the process $(X(t))_{t\geq 0}$ where $X(t)$ is the number of customers in the system (while waiting or while being served) at time t.

1) Show that $(X(t))_{t\geq 0}$ is a Markov process (homogenous).

2) Compute the generator of the process.

3) Find the transition matrix of the jumps and deduce again the generator of the chain.

4) Determine the probability distribution of the time that the chain spends in each state.

5) Compute the asymptotic distribution of the number of customer in the queue.

6) Discuss the nature of the states with respect to the parameters μ and λ.

Exercise 3. $M/M/1/m$ queue
We consider again the queuing system of last exercise, except that the waiting room as a maximum capacity of $m - 1$ customers. So that if the system starts with less than m clients, the number of clients in the system will never be bigger than m, because a client who arrives while m clients are already in the system goes away and never comes back. Nevertheless it is possible that the initial state is bigger than m.

1) Compute the generator and the transition matrix of the Markov chain for the jumps.

2) Determine the nature of the state for the associated Markov chain.

3) Compute the asymptotic distribution of the number of clients in the queue.

Exercise 4. $M/M/\infty$ queue
Let us assume that the arrivals time of clients in a system follow a Poisson process with parameter λ. The system is built from a countable collection of servers so that when a client arrives, his service starts immediately.
We assume that the service time follows an exponential distribution with parameter μ, the service duration of one customer is independent from the one of the others and independent from the Poisson process of arrivals.
let us consider the process $(X(t))_{t\geq 0}$ where $X(t)$ is the number of clients in the system at time t.

1) Compute the generator and the transition matrix of the Markov chain fo the jumps.

2) Compute the asymptotic distribution of the number of clients in the queue.

Exercise 5. $M/M/m/\infty$ queue
Let us assume that the arrivals time of clients in a system follow a Poisson process with parameter λ. The system is built from m servers and a waiting room of infinite capacity (when a client arrives, his service starts immediately if one of the m desk is free, otherwise he goes in the waiting room).
The service duration of one customer is independent from the one of the others and independent from the Poisson process of arrivals.
Consider the process $(X(t))_{t\geq 0}$ where $X(t)$ is the number of clients in the system at time t.

1) Compute the generator and the transition matrix of the Markov chain fo the jumps.

2) Compute the asymptotic distribution of the number of clients in the queue.

Exercise 6. Let us assume that the arrivals time of clients in a system follow a Poisson process with parameter λ. The system is built from one server and a waiting room of infinite capacity (when a client arrives, his service starts immediately if the desk is free, otherwise he goes in the waiting room).
We assume that the service time follows an exponential distribution with parameter μ, the service duration of one customer is independent from the one of to the others and independent from the Poisson process of arrivals.
Clients are busy: a client that cannot be served immediately waits a random time exponentially distributed with parameter γ, then if his service has not already started he goes away and never comes back (there s however no restriction on the duration of service).
Consider the process $(X(t))_{t\geq 0}$ where $X(t)$ is the number of clients in the system at time t.

1) Compute the generator and the transition matrix of the Markov chain fo the jumps.

2) Gives the stationary distribution for every state $i > 0$ of this process in term of the stationary probability that the system is void.